TRANSFORMATIONS IN LINEAR TOPOLOGICAL SPACES

By John V. Wehausen

A DISSERTATION

Submitted in Partial Fulfillment of the Requirements
for the Degree of Doctor of Philosophy in the
University of Michigan

Reprinted from Duke Mathematical Journal
Vol. 4, No. 1, March, 1938

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BY JOHN V. WEHAUSEN

1. **Introduction.** In this paper some well known theorems for linear metric spaces whose proofs do not involve completeness are extended to linear topological spaces. The relation between bounded sets and linear continuous transformations is considered; and, in connection with this, metrizability conditions for linear topological spaces are obtained in terms of boundedness. A linear continuous functional on a linear topological space is shown to operate essentially on only a "part" of the space which can be normed. Finally the relation between completeness and category is considered and a theorem is proved which shows that two definitions of completeness which have been given do not imply that the space is of the second category.

2. **Topologies for linear spaces.** In the postulates for a linear set [cf. **1**, p. 26][1] there is no notion of a topology. If a topology is imposed on a linear set in such a fashion that the postulated operations of addition of elements and multiplication of elements by real numbers are continuous in the topology, the linear set will be said to be a linear topological space [cf. **2**, pp. 201–204]. The most important topologies of this type for linear sets have been metric topologies, viz., the F-metric [**1**, p. 35] and the norm [**1**, p. 53].[2] As a matter of fact, it will be shown that the F-metric is the most general metric of this nature.

However, two non-metric topologies have been given in the literature. Kolmogoroff [**3**, p. 29] has simply postulated an operation of closure, $\bar{M}$, for any set M satisfying the Riesz-Kuratowski axioms, viz.,

(1) if M is a single element, $\bar{M} = M$,
(2) $\bar{\bar{M}} = \bar{M}$,
(3) $\overline{M + N} = \bar{M} + \bar{N}$,

and then in addition required that $x + y$ and αx be continuous in the topology. He proves that such a space is a regular Hausdorff space. On the other hand, John von Neumann [**4**, p. 4] has defined a topology for a linear set L by means of a set of neighborhoods $\mathfrak{U} = \{U\}$ satisfying the following axioms:

(1) $\theta \in U$ for every $U \in \mathfrak{U}$.
(2) If $U, V \in \mathfrak{U}$, there exists $W \in \mathfrak{U}$ such that $W \subset \mathfrak{P}(U, V)$.
(3) If $U \in \mathfrak{U}$, there exists $V \in \mathfrak{U}$ such that $V + V \subset U$.
(4) If $U \in \mathfrak{U}$, there exists $V \in \mathfrak{U}$ such that $\alpha V \subset U$ for $|\alpha| \leqq 1$.

Received November 9, 1937.

[1] Boldface numbers refer to the bibliography at the end of the paper.

[2] Other non-continuous metrics have been introduced by C. R. Adams [**12**, pp. 422, 423].

(5) If $x \in L$, $U \in \mathfrak{U}$, there exists α such that $x \in \alpha U$.

L is called a convex space if $\mathfrak{U}$ satisfies in addition

(6) $U + U \subset 2U$ for all $U \in \mathfrak{U}$.

The neighborhood system for any element is then obtained by translating the neighborhoods for θ. It is proved by von Neumann that Axioms (1)-(5) give a regular Hausdorff space and that the operations of addition and multiplication are continuous [**4**, p. 6, Theorems 6 and 7]. This topology then satisfies Kolmogoroff's axioms. The converse is also true. As the neighborhood system for each element x of the space take the set of all open sets containing the element, denoting it by $\mathfrak{U}(x)$. First it must be shown that the topology is "uniform" over the whole space, i.e., if $x \in L$, then the translated neighborhood system $\mathfrak{U}(\theta) + x$ is equivalent to $\mathfrak{U}(x)$. Let $U_x \in \mathfrak{U}(x)$ be arbitrary. Since $\theta + x = x$, it follows from continuity of addition that there exist $V_\theta \in \mathfrak{U}(\theta)$ and $V_x \in \mathfrak{U}(x)$ such that $V_\theta + V_x \subset U_x$. Since $x \in V_x$, $V_\theta + x \subset U_x$. The definition of continuity in connection with the equation $x + (-x) = \theta$ gives the other half of the equivalence. Therefore the topology of Kolmogoroff is uniform. It then only remains to show that Axioms (1)-(5) are satisfied by $\mathfrak{U}(\theta)$. But (1) is satisfied by definition of $\mathfrak{U}(\theta)$ and (2) is a property of any Hausdorff space. Axiom (3) follows from continuity of addition as applied to $\theta + \theta = \theta$ and (4) from continuity of multiplication as applied to $0 \cdot \theta = \theta$. Consider now (5). Let $x \in L$ and $U \in \mathfrak{U}(\theta)$ be arbitrary and suppose that x non-$\in nU$, i.e., $n^{-1}x$ non-$\in U$ for every integer n. But since multiplication is continuous $n^{-1}x \to \theta$, i.e., there exists $n(U)$ such that, for $n \geqq n(U)$, $n^{-1}x \in U$. This is a contradiction. The following theorem has thus been established.

THEOREM 1. *The topologies of Kolmogoroff and von Neumann are equivalent.*

Tychonoff [**6**, p. 768] has called a linear topological space satisfying Kolmogoroff's axioms "locally convex" if for every U_x there exists a convex[3] neighborhood V_x such that $V_x \subset U_x$. It follows easily from the regularity of the space and a theorem of von Neumann [**4**, p. 10, Theorem 13] that the notions of "convex" and "locally convex" space are the same. One may therefore state the following corollary of the previous theorem.

COROLLARY. *The convex topologies of von Neumann are equivalent to the locally convex topologies of Tychonoff.*

3. **Preliminary definitions.** If L_1 and L_2 are two linear topological spaces, the definitions of additive and homogeneous transformations on L_1 to L_2 are unchanged. Continuity of transformations is defined as usual for neighborhood spaces. It is easily seen that an additive transformation which is continuous at one point is everywhere continuous and is homogeneous, so that one need define continuity for such transformations only at the zero-element, θ, i.e., if $\mathfrak{U} = \{U\}$ and $\mathfrak{V} = \{V\}$ are the neighborhood systems in L_1 and L_2 respectively, then $T(x)$ on L_1 to L_2 is continuous if and only if for arbitrary $V \in \mathfrak{V}$ there exists $U \in \mathfrak{U}$ such that $T(U) \subset V$.

[3] A set M is convex if, whenever $x, y \in M$, $\alpha, \beta \geqq 0$ and $\alpha + \beta = 1$, then $\alpha x + \beta y \in M$.

Mazur and Orlicz [7, p. 152] have used the following definition for a bounded set:[4] $R \subset L$ is bounded if for every sequence $\{x_i\} \subset R$ and every sequence of real numbers $\alpha_i \to 0$, the sequence $\alpha_i x_i \to \theta$. On the other hand, von Neumann [4, p. 7, Definition 5] uses this definition: $R \subset L$ is bounded if for every $U \in \mathfrak{U}$ there exists a real number α such that $R \subset \alpha U$. The following lemma is stated without proof.

LEMMA. *The definitions of boundedness of Banach and of von Neumann are equivalent.*[5]

4. Relation between bounded sets and continuity of additive transformations. In normed linear spaces the equivalence of the class of additive transformations taking bounded sets into bounded sets and the class of additive continuous transformations is a well known theorem. This theorem has been extended by Mazur and Orlicz [7, p. 153] to F-spaces. Half of the equivalence may be easily proved for linear topological spaces also.

THEOREM 2. *If $T(x)$ is a linear continuous transformation on L_1 to L_2 and if $R \subset L_1$ is bounded, then $T(R) \subset L_2$ is bounded.*

For let $V \in \mathfrak{V}$ be arbitrary. Then by continuity of $T(x)$ there exists $U \in \mathfrak{U}$ such that $T(U) \subset V$. Since R is bounded, there exists α such that $R \subset \alpha U$. Hence, $T(R) \subset T(\alpha U) = \alpha T(U) \subset \alpha V$, i.e., $T(R)$ is bounded since V was arbitrary.

The converse of this theorem may be proved in the following restricted form.

THEOREM 3. *If L contains a bounded open set and if the additive transformation $T(x)$ takes bounded sets into bounded sets, $T(x)$ is continuous.*

The proof will not be given since a somewhat more general theorem will be proved later.

5. Two metrizability conditions. The requirement in the preceding theorem that there exist a bounded open set in the domain is a considerable restriction as the following theorem shows.

THEOREM 4. *A linear topological space is metrizable as an F-space if there exists a bounded open set in the space.*

Suppose G is a bounded open set. One may assume $\theta \in G$ for otherwise one would simply consider the set $G_x = G - x$, where $x \in G$, also a bounded open set. First note that for any real α the set αG is open. Next, that the set of open sets $\{n^{-1}G\}$ is equivalent to $\mathfrak{U} = \{U\}$. For, since $n^{-1}G$ is open, one can find for any n a neighborhood $U(n) \in \mathfrak{U}$ such that $U(n) \subset n^{-1}G$. On the other hand, let $U \in \mathfrak{U}$ be arbitrary. Then, by Axiom 3 of von Neumann, there exists $V \in \mathfrak{U}$ such that $\beta V \subset U$ for $|\beta| \leqq 1$. Since G is bounded there exists α such that $G \subset \alpha V$. Now select some integer $n \geqq |\alpha|$ so that $|n^{-1}\alpha| \leqq 1$. Then

[4] The definition is used by Mazur and Orlicz for F-spaces. Kolmogoroff [3, p. 30] has used it for topological spaces. It is originally due to Banach.

[5] This lemma has been stated by D. H. Hyers, Bull. Amer. Math. Soc., vol. 43 (1937), p. 203, Abstract 228.

$n^{-1}G \subset n^{-1}\alpha V \subset U$. This proves the equivalence. The space then satisfies the first denumerability axiom of Hausdorff. One may now use the result of Garrett Birkhoff [8, p. 428] that a Hausdorff group is metrizable if and only if it satisfies the first denumerability axiom. Since a linear space satisfying the axioms of von Neumann is a Hausdorff group with respect to addition, the result applies here also. To show that the metric is an F-metric one must show (1) $\rho(x, y) = \rho(x - y, \theta)$, (2) $\alpha_n \to 0$, $x \in L$ imply $\alpha_n x \to \theta$, (3) α real, $x_n \to \theta$ imply $\alpha x_n \to \theta$. (1) follows immediately from the fashion in which Birkhoff defines his metric. One may show (2) and (3) independently of metrizability considerations. In fact, (2) and (3) are immediate consequences of the continuity of multiplication. This completes the proof.[6]

The converse of this theorem is not true, i.e., there exist F-spaces in which no sphere is bounded. This is true, for example, of the spaces (S) and (s) [1, pp. 9, 10].

In the last part of the proof of the preceding theorem the following theorem was also proved.

THEOREM 5. *A linear topological space is metrizable as an F-space if and only if it satisfies the first denumerability axiom.*[7]

From this theorem and from Kolmogoroff's axioms for a linear topological space the following corollary is easily obtained.

COROLLARY. *Any linear metric space in which the operations of multiplication and addition are continuous may be metrized with an equivalent F-metric.*

In case the space is convex, i.e., satisfies Axiom (6), one may prove the following theorem.

THEOREM 6. *A necessary and sufficient condition that a convex linear topological space have an equivalent norm-metric is that it contain a bounded open set.*

If G is a bounded open set, one may assume as in the preceding theorem that $\theta \in G$. Following the notation of von Neumann, let

$$G_{\text{conv}} \equiv E_x\left[x = \sum_{i=1}^{n} \alpha_i x_i, \quad \alpha_i \geqq 0, \quad \sum_{i=1}^{n} \alpha_i = 1, \quad x_i \in G\right].$$

LEMMA 1. *If L is a convex space, the boundedness of G is both necessary and sufficient for the boundedness of G_{conv}.*

This has been proved by von Neumann [4, p. 10, Theorem 14].

LEMMA 2. *If G is an open set, then also G_{conv} is open.*

Let $x = \sum_{k=1}^{n} \alpha_k x_k$ be any element of G_{conv} where $x_k \in G$. Since G is open, there exist neighborhoods U_k such that $x_k + U_k \subset G$. From Axiom (2) one easily

[6] An F-space is generally required to be also complete. This has not been done here. However, sequential completeness, a notion which will be defined later, is sufficient to obtain completeness in the metric sense.

This theorem has been stated by D. H. Hyers, loc. cit.

deduces the existence of $U \in \mathfrak{U}$ such that $U \subset \mathfrak{P}(U_1, \cdots, U_n)$. Then $x_k + U \subset G$ for $k = 1, \cdots, n$. Hence

$$\alpha_1(x_1 + U) + \cdots + \alpha_n(x_n + U) = \alpha_1 x_1 + \cdots + \alpha_n x_n + \alpha_1 U + \cdots + \alpha_n U \subset G_{\text{conv}}.$$

Since $\alpha_1 U + \cdots + \alpha_n U \supset (\alpha_1 + \cdots + \alpha_n)U = U$, one sees that $\alpha_1 x_1 + \cdots + \alpha_n x_n + U \subset G_{\text{conv}}$, i.e., G_{conv} is open.

It now follows from these two lemmas that G_{conv} is a bounded, open, convex set. The following theorem of Kolmogoroff's [3, p. 30] may now be applied to obtain the theorem to be proved: *A topological linear space has an equivalent norm topology if and only if there exists a bounded open convex set in the space.*

6. **Extension of Theorem 3.** Theorem 4 shows that the conditions for which Theorem 3 has been stated are rather stringent since it allows for the domain of the transformation $T(x)$ only a proper subclass of the class of F-spaces. This is too restrictive, for wherever the domain is any F-space and the range is any linear topological space, it is easy to obtain the converse of Theorem 2. The method of proof is the same as that used by Mazur and Orlicz [6, p. 153].

THEOREM 3′. *If L_1 is an F-space and L_2 a linear topological space, an additive transformation $T(x)$ is continuous if it takes bounded sets into bounded sets.*

Suppose $x_n \to \theta$, i.e., $|x_n| = \rho(x_n, \theta) \to 0$. Then there exists a sequence of integers $k_n \to \infty$ such that $k_n |x_n| \to 0$. Since $k_n |x_n| \geqq |k_n x_n|$, $k_n x_n \to \theta$ and therefore $\{k_n x_n\}$ is bounded. Hence $\{T(k_n x_n)\}$ is bounded, so that $(k_n)^{-1} T(k_n x_n) = T(x_n) \to \theta$. Now suppose there exists $V_0 \in \mathfrak{V}$ such that for each sphere $S_n = E_x[|x| < n^{-1}]$ there exists $x_n \in S_n$ such that $T(x_n)$ non-$\in V_0$. Then $|x_n| \to 0$, i.e., $x_n \to \theta$. Hence $T(x_n) \to \theta$ so that there must exist n_0 such that, for $n \geqq n_0$, $T(x_n) \in V_0$. This is a contradiction.

7. **Strongly bounded transformations.** An additive transformation $T(x)$ will be called strongly bounded if for every $U \in \mathfrak{U}$ and $V \in \mathfrak{V}$ it is possible to find α such that $T(U) \subset \alpha V$, i.e., for each $U \in \mathfrak{U}$, $T(U)$ is a bounded set. The following theorem is then true.

THEOREM 7. *An additive transformation $T(x)$ which is strongly bounded is continuous.*

Let $V \in \mathfrak{V}$ be arbitrary; also $U \in \mathfrak{U}$. By Axiom 3 there exists $V_1 \in \mathfrak{V}$ such that $\alpha V_1 \subset V$ for $|\alpha| \leqq 1$. By hypothesis there exists β such that $T(U) \subset \beta V_1$. Let k be any integer such that $|k^{-1}\beta| \leqq 1$. Then there exists $U_1 \in \mathfrak{U}$ such that $kU_1 \subset U$. Hence $T(kU_1) \subset T(U) \subset \beta V_1$, or $T(U_1) \subset k^{-1}\beta V_1 \subset V$, i.e., $T(x)$ is continuous.

The converse of this theorem is true provided the neighborhoods in the domain are bounded sets. This would imply, however, that the domain is metrizable as an F-space in which spheres are bounded sets.

8. **Linear transformations in convex spaces.** In a convex space pseudo-norms have been defined by von Neumann [**4**, pp. 18, 19] as follows. Let $|| x ||_U^+$ equal the greatest lower bound of $\alpha > 0$ such that $x \in \alpha U$ and let $|| x ||_U = \max (|| x ||_U^+, || - x ||_U^+)$. He has shown that $|| x ||_U$ has the following properties: $|| x + y ||_U \leqq || x ||_U + || y ||_U$, $|| \alpha x ||_U = | \alpha | \cdot || x ||_U$, $|| \theta ||_U = 0$, $|| x ||_U$ is a continuous function of x. The pseudo-norm $|| x ||_U$ lacks the norm property $|| x || = 0$ if and only if $x = \theta$. However, by use of pseudo-norms, one may obtain generalizations of several well known theorems for normed spaces. An existence theorem will be proved first.

THEOREM 8. *If L is a convex space, then for any $x_0 \in L$ and any $U \in \mathfrak{U}$ there exists a linear continuous functional $F(x)$ defined on L with the property that $F(x_0) = || x_0 ||_U$.*

Denote by K the linear set $E_x[\alpha x_0$, α real] and define on K the functional $f(\alpha x_0) = \alpha || x_0 ||_U$. Then $f(x)$ is obviously linear on K. Then, if we denote $|| x ||_U$ by $p(x)$, it follows that $p(x + y) \leqq p(x) + p(y)$, $p(tx) = tp(x)$ for $t \geqq 0$ and $p(x) \geqq f(x)$ for $x \in K$. One may now use a theorem of Banach's on the extension of additive functionals [**1**, p. 27, Theorem 1] and assert that there exists a linear functional $F(x)$ defined on all L such that $F(x) = f(x)$ for $x \in K$ and $-|| x ||_U \leqq F(x) \leqq || x ||_U$ for all x. Since $|| x ||_U$ is continuous at θ, it follows that $F(x)$ is also continuous at θ and hence, since it is additive, that it is continuous.

By the use of pseudo-norms a bounded, additive transformation may be defined in a manner analogous to the definition in a normed space. An additive transformation $T(x)$ on L_1 to L_2, L_1 and L_2 convex spaces, will be said to be *bounded* if to each $V \in \mathfrak{V}$ correspond $U \in \mathfrak{U}$ and a real number $M(U, V) \geqq 0$ such that $|| T(x) ||_V \leqq M(U, V) || x ||_U$. Such a transformation obviously takes bounded sets into bounded sets. The following theorem connecting bounded and continuous additive transformations is proved.

THEOREM 9. *An additive transformation $T(x)$ is continuous if and only if it is bounded.*

If $T(x)$ is continuous, for each $V \in \mathfrak{V}$ one may find $U \in \mathfrak{U}$ such that $T(U) \subset V$, or, using pseudo-norms, such that $|| x ||_U < 1$ implies $|| T(x) || < 1$. Note that if $|| x ||_U = 0$, then $|| T(x) ||_V = 0$ for otherwise one could find a real α such that $|| T(\alpha x) ||_V = | \alpha | \cdot || T(x) ||_V > 1$ even though $|| \alpha x ||_U = 0$. Suppose now that the required $M(U, V)$ does not exist. Select a sequence of real positive integers $M_n \to \infty$ and assume that to each n corresponds an x_n such that $|| T(x_n) ||_V > M_n || x_n ||_U \neq 0$. Let

$$y_n = \frac{x_n}{M_n || x_n ||_U}.$$

Then

$$|| T(y_n) ||_V = \frac{1}{M_n || x_n ||_U} || T(x_n) ||_V > 1.$$

But $|| y_n ||_U = M_n^{-1} \to 0$, so that one can find n_0 such that, for $n \geqq n_0$, $|| y_n ||_U < 1$ so that $|| T(y_n) ||_V < 1$, a contradiction for $n \geqq n_0$.

On the other hand, suppose $T(x)$ is bounded. Then for arbitrary $V \in \mathfrak{V}$ one may find $U \in \mathfrak{U}$ and $M(U, V) \geqq 0$ such that $|| T(x) ||_V \leqq M(U, V) \cdot || x ||_U$. Then $M(U, V) || x ||_U < 1$ or $x \in [M(U, V)]^{-1}U$ implies that $T(x) \in V$. Since there exists $U_1 \subset [M(U, V)]^{-1}U$, it follows that $T(U_1) \subset V$, i.e., $T(x)$ is continuous.

The following corollary is an immediate consequence.

COROLLARY. *If L is convex and if $F(x)$ is an additive functional defined on L, $F(x)$ is continuous if and only if there exist $U \in \mathfrak{U}$ and $M(U) \geqq 0$ such that $| F(x) | \leqq M(U) || x ||_U$.*

In connection with this corollary the following theorem may be proved.

THEOREM 10. *Suppose a linear functional $F(x)$ on a convex space L is continuous if and only if for every $U \in \mathfrak{U}$ there exists $M(U) \geqq 0$ such that $| F(x) | \leqq M(U) || x ||_U$. Then any of the pseudo-norms $|| x ||_U$ for $U \in \mathfrak{U}$ is also a norm and the corresponding norm topology is not stronger[8] than the given topology but is equivalent to it with respect to the determination of the class of linear and continuous functionals.*

Since pseudo-norms have all the norm properties except $|| x ||_U = 0$ implies $x = \theta$, only this need be shown. Suppose $|| x_0 ||_U = 0$ for some $U \in \mathfrak{U}$. First note that there exists $U_0 \in \mathfrak{U}$ such that $|| x_0 ||_{U_0} \neq 0$ unless $x_0 = \theta$. Then, by Theorem 8, there exists a linear continuous functional $F_0(x)$ such that $F_0(x_0) = || x_0 ||_{U_0} \neq 0$. On the other hand, since $| F(x) | \leqq M(U, F) || x ||_U$, $F(x_0) = 0$ for all linear continuous F. Hence $F_0(x_0) = 0$ which is a contradiction unless $x_0 = \theta$. Therefore $|| x ||_U = 0$ if and only if $x = \theta$, i.e., $|| x ||_U$ is a true norm. If one now fixes $U_0 \in \mathfrak{U}$ one may define the "spherical" neighborhoods in the corresponding norm topology, viz., $S_\alpha = E_x[|| x ||_{U_0} < \alpha]$. From the fact that for any α there exists U_1 such that $U_1 \subset \alpha U_0$ it follows that any sphere S_α contains a neighborhood of $\mathfrak{U}$, i.e., the norm topology is not stronger than the given topology. From the hypothesis of the theorem any linear functional which is continuous in the neighborhood topology is continuous in the norm topology. On the other hand, since any linear functional which is continuous in the norm topology is bounded, i.e., there exists $M \geqq 0$ such that $| F(x) | \leqq M || x ||_U$, it follows from the corollary to Theorem 9 that $F(x)$ is continuous in the neighborhood topology.

The following theorems are generalizations of well known theorems for normed spaces [**1**, p. 55, Theorem 4; p. 57, Lemma; p. 58, Theorem 6]. The proofs will not be given since they are quite similar to those for the corresponding theorems in normed spaces.

THEOREM 11. *If $f(x)$ is any functional defined on G, a subset of a convex*

[8] One topology will be said to be not stronger than another if every limit point in the sense of the first is also a limit point in the sense of the second.

space L, there exists a linear continuous functional $F(x)$ defined on all L such that

(1) $F(x) = f(x)$ *for* $x \in G$,

(2) $|F(x)| \leqq M \|x\|_U$ *for given* $M \geqq 0$ *and* $U \in \mathfrak{U}$ *if and only if for all finite sets* $\{x_i\} \subset G$ *and real numbers* $\{\alpha_i\}$

$$\left|\sum_{i=1}^{n} \alpha_i f(x_i)\right| \leqq M \cdot \left\|\sum_{i=1}^{n} \alpha_i x_i\right\|_U .$$

THEOREM 12. *If $G \subset L$ is a linear set, $U \in \mathfrak{U}$ any neighborhood, and $y_0 \in L$ an element such that, for $x \in G$, $\|y_0 - x\|_U \geqq d > 0$, there exists a linear continuous functional $F(x)$ defined on L such that*

(1) $F(y_0) = 1$,

(2) $F(G) = 0$,

(3) $|F(x)| \leqq d^{-1} \|x\|_U$. *If d is the greatest lower bound of the values $\|y_0 - x\|_U$ for $x \in G$, then in (3) d^{-1} is the smallest value M such that $|F(x)| \leqq M \|x\|_U$.*

THEOREM 13. *If $G \subset L$ is an arbitrary set and if $y_0 \in L$, a necessary and sufficient condition that y_0 belong to the closed linear extension of G is that $F(x) = 0$ for $x \in G$ imply $F(y_0) = 0$ for all linear continuous $F(x)$.*

9. **On the domain of a linear continuous functional.** In order to provide a motivation for and examples of the succeeding theorem the general form of the linear continuous functional on three F-spaces is given.

Space (s). The general form of the linear continuous functional on (s) is $F(x) = \sum_{i=1}^{n} \varphi_i \xi_i$, where n and $\varphi_1, \cdots, \varphi_n$ depend on F and where $x = (\xi_1, \xi_2, \cdots)$ [**1**, p. 50, Theorem 11].

Space (S). The most general linear functional on (S) is the zero-functional [**1**, p. 234].

Denote by (CF) the set of all continuous functions defined on $[-\infty, \infty]$ with the following metric: $\rho(x, y) = \sum_{n=1}^{\infty} \frac{(x, y)_n}{1 + (x, y)_n}$, where $(x, y)_n = \max_{|t| \leqq n} |x(t) - y(t)|$ [cf. **9**, pp. 30, 31].

Space (CF). Every linear continuous functional on (CF) is of the form $F(x) = \int_{\alpha}^{\beta} x(t)\, d\varphi(t)$, where α, β and $\varphi(t)$ depend on F and $\varphi(t)$ is of bounded variation in the interval $[\alpha, \beta]$.

Since the proof of this involves no unusual procedure, it is not given here.

In each of these examples the functional seems to operate essentially only on a "part" of each element of the space in such a manner that when a set of elements converges according to the F-metric, their corresponding "parts" converge according to a norm-metric. The precise meaning of this is given by the next theorem. Before stating this theorem a definition will be given.

DEFINITION. A set $X \subset L$ will be called a *flat subspace* if for arbitrary $x \in X$, $X - x$ forms a linear set.

THEOREM 14. *If $F(x)$ is a linear continuous functional on a linear topological space L, there corresponds to each $x \in L$ a flat subspace X such that*

(1) *if $x_1, x_2 \in X$, then $F(x_1) = F(x_2) \equiv F(X)$,*

(2) *there exists a norm topology, $\| X \|_F$, for the set, L_F, of such subspaces such that if $x \in \bar{R}$, $X \in \bar{R}_F$,*

(3) *if $\| X_n - X \|_F \to 0$, then $F(X_n) \to F(X)$,*

(4) *the linear continuous functionals on L_F determined by $\| X \|_F$ are linear continuous functionals on L with $F(x) \equiv F(X)$ for $x \in X$.*

Define $X \equiv E_x[x = \bar{x} + x', \text{ where } F(x) = F(\bar{x})]$. Each $x \in L$ is obviously in some X. Letting $\Theta = E_x[F(x) = 0]$, we easily see that Θ is a linear set. Since $X = \Theta + \bar{x}$, it follows that X is a flat subspace. It also easily follows from the linearity of L and of F that L_F is a linear set.

(1) is an obvious consequence of the definition of X.

Consider (2). A topology will now be defined for L_F. If R is a set in L, the set of elements in L_F corresponding to the elements of R will be denoted by R_F. The closure of a set R_F will be defined as follows: $X \in \bar{R}_F$ if $F(X) \in \overline{F(R_F)}$. Although $x \in \bar{R}$ does not imply that x is a sequential limit point of R, the topology in L_F will obviously be of a sequential type. This topology also satisfies Kolmogoroff's axioms for a linear topological space, the continuity of addition and multiplication following from the linearity and continuity of $F(x)$. Now let $S_\alpha = E_X[| F(X) | < \alpha]$. It then follows from the linearity of $F(x)$ that S_α is a bounded, open, and convex subset of L_F. The normability condition of Kolmogoroff used in the proof of Theorem 6 may now be applied to give an equivalent norm topology for L which will be denoted by $\| X \|_F$. Then, if $x_0 \in \bar{R}$, it follows from continuity of $F(x)$ that $F(x_0) \in \overline{F(R)}$ and therefore $F(X_0) \in \overline{F(R_F)}$, i.e., $X_0 \in \bar{R}_F$.

(3) follows immediately from the definition of $\| X \|_F$.

In order to prove (4) let $\mathfrak{F}_0$ be the set of all linear continuous functionals on L_{F_0} determined by $\| X \|_{F_0}$. If $F \in \mathfrak{F}_0$, define $F(x) = F(X)$ for $x \in X$. Then, if $x_0 \in \bar{R}$, $X_0 \in \bar{R}_{F_0}$ by (2). By continuity of $F(X)$, $F(X_0) \in \overline{F(R_{F_0})}$ and hence $F(x_0) \in \overline{F(R)}$, i.e., $F(x)$ is continuous.

10. **Category of linear topological spaces.** In the study of complete metric spaces one of the fundamental theorems as far as the theory of functions for such spaces is concerned is the one which states that every such space is of the second category. Since this property seems to be important in the proofs of many theorems, one should like a definition of completeness for topological linear spaces to give the same or a corresponding theorem. J. von Neumann has shown how the ordinary notion of "sequential" completeness can be carried over into such spaces and, then, in order to avoid a certain difficulty in connection with it, proposes to call a topological linear space "topologically" complete if every closed and totally bounded set is compact [**4**, pp. 1–3]. The following theorem shows that for neither of these definitions the category theorem holds.

In order to state the theorem the weak neighborhood topology for a normed space L must be defined. For each finite set of linear continuous functionals, $F_1, \cdots, F_n$, defined on L and for each $\delta > 0$, let the neighborhood

$$U(F_1, \cdots, F_n; \delta) \equiv E_x[|F_i(x)| < \delta, i = 1, \cdots, n].$$

The set of all such neighborhoods will be denoted by $\mathfrak{U}$ and defines the weak topology[9] for L. These neighborhoods satisfy Axioms (1)–(6), i.e., the space is a convex linear topological space. The theorem is then the following.

THEOREM 15. *A normed space with its weak neighborhood topology for which there exists no finite total set of functionals is of the first category.*

The proof will be broken up into several lemmas.

LEMMA 1. *If no neighborhood of a topological linear space L is bounded, every bounded set is non-dense.*

If a set $M \subset L$ is bounded, then also its closure $\bar{M}$ is bounded [**4**, p. 9, Theorem 11]. Since no neighborhood is bounded, it would be impossible to find for any $x \in \bar{M}$ a $U \in \mathfrak{U}$ such that $x + U \subset \bar{M}$, i.e., $\bar{M}$ has a vacuous interior. But this is just the condition that M be non-dense [**10**, p. 31, II].

LEMMA 2. *The set $S_n = E_x[\|x\| \leqq n]$ is bounded.*

Consider arbitrary $U(F_1, \cdots, F_k; \delta) \in \mathfrak{U}$. Let $m = \max(\|F_1\|, \cdots, \|F_k\|)$ and let $\alpha = 2mn\delta^{-1}$. Then $\alpha U(F_1, \cdots, F_k; \delta) = E_x[|F_i(x)| < \alpha\delta = 2mn]$. Now suppose $x \in S_n$. Then $|F_i(x)| \leqq \|F_i\| \cdot \|x\| \leqq mn < 2mn$, i.e., $x \in \alpha U$. Therefore $S_n \subset \alpha U(F_1, \cdots, F_k; \delta)$ and S_n is bounded since U was arbitrary.

LEMMA 3. *If L does not have a finite total set of functionals, then none of its weak neighborhoods are bounded.*

Consider arbitrary $U(F_1, \cdots, F_n; \delta) \in \mathfrak{U}$. Let $G = E_x[F_i(x) = 0, i = 1, \cdots, n]$. Since the set, $F_1, \cdots, F_n$, cannot be total, G must contain other elements than θ. Then, since G is a linear set, one can find $x_0 \in G$ with $\|x_0\| = 1$. Also, by a theorem of Banach's [**1**, p. 55, Theorem 3], there exists a linear continuous functional $F_0(x)$ such that $F_0(x_0) = \|x_0\| = 1$. Now consider the neighborhood $U(F_0; 1) = E_x[|F_0(x)| < 1]$ and the sequence $x_0, 2x_0, \cdots, kx_0, \cdots$. Since $kx_0 \in G$, every element of the sequence is contained in $U(F_1, \cdots, F_n; \delta)$. However, since $mU(F_0; 1) = E_x[|F_0(x)| < m]$, one easily sees that, for $k \geqq m$, kx_0 non-$\in mU(F_0; 1)$. It is then obvious that there exists no number α such that $U(F_1, \cdots, F_n; \delta) \subset \alpha U(F_0; 1)$, i.e., $U(F_1, \cdots, F_n; \delta)$ which was arbitrary is not bounded.

From the three lemmas it follows that the set S_r is non-dense. Then, since $L = \sum_{n=1}^{\infty} S_n$, it is of the first category in itself.

Not all "weak" normed spaces are sequentially complete, an important

[9] Note that the weak neighborhood topology is not necessarily equivalent to the weak sequential topology for L (i.e., where $x_k \to x$ is defined as meaning $F(x_k) \to F(x)$ for all linear continuous F). In fact, in Hilbert space there exist points which are limit points of sets without being sequential limit points [**5**, p. 380]. The two topologies are equivalent, however, as far as convergence of sequences is concerned.

example being the space (C) of all continuous functions defined on the interval $[0, 1]$. However, the spaces L_p and l_p for $p \geqq 1$ are sequentially complete in their weak topologies [**1**, pp. 140–143, §4].[10] J. von Neumann [**4**, p. 17, Theorem 23] has shown that abstract Hilbert space with its weak neighborhood topology is also topologically complete. Therefore neither definition is sufficient to obtain the property that the space be of the second category. The additional remark should be made that none of the spaces L_p, l_p or abstract Hilbert space have finite total sets of functionals defined on them.

The extent of the restriction in the preceding theorem that the normed space have no finite total set of functionals is given by the following theorem.

THEOREM 16. *A normed space L has a finite total set of functionals defined on it if and only if it is finite dimensional.*

Suppose $F_1, \cdots, F_n$ is a total set of functionals for L and assume that for every k there exist $z_1, \cdots, z_k \subset L$ such that $\sum_{i=1}^{k} \alpha_i z_i = \theta$ implies $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$. Now consider the set of equations

$$\sum_{j=1}^{m} \beta_j F_i(z_j) = 0 \qquad (i = 1, \cdots, n),$$

where the β_j are to be the unknowns. Then, if $m > n$, it is obvious that the rank of the matrix of the coefficients, $(F_i(z_j))$, cannot exceed n. By a well known theorem on solutions of linear homogeneous equations, there exist at least $m - n > 0$ linearly independent solutions of the set of equations, i.e., there exists $z = \sum_{j=1}^{m} \beta_j z_j \neq \theta$ such that $F_i\left(\sum_{j=1}^{m} \beta_j z_j\right) = 0$ for $i = 1, \cdots, n$. But, by the totality of the F_i, one must have $\sum_{j=1}^{m} \beta_j z_j = \theta$ which is a contradiction. Therefore the space is at most n-dimensional.

Suppose L is n-dimensional, i.e., there exist $z_1, \cdots, z_n$, linearly independent, such that to each $x \in L$ there corresponds a set of real numbers $\alpha_1, \cdots, \alpha_n$ with the property that $x = \sum_{i=1}^{n} \alpha_i z_i$. Let G_i be the linear manifold determined by $z_1, \cdots, z_{i-1}, z_{i+1}, \cdots, z_n$. Then there exists a linear continuous functional $F_i(x)$ such that $F_i(z_i) = 1$ and $F_i(G_i) = 0$ [**1**, p. 57, Lemma]. Then $F_1, \cdots, F_n$ is a total set. For suppose that, for $x = \sum_{j=1}^{n} \alpha_j z_j$, $F_i(x) = 0$ $(i = 1, \cdots, n)$. Then $F_i\left(\sum_{j=1}^{n} \alpha_j z_j\right) = \sum_{j=1}^{n} \alpha_j F_i(z_j) = \alpha_i = 0$, i.e., $x = \theta$.

One may now obtain some additional information concerning the metrizability of the weak neighborhood topologies, viz., the following theorem.

[10] Since this is a sequential notion it makes no difference that the "sequential" and "neighborhood" weak topologies are not equivalent as long as the "sequential part" of the latter is equivalent to the former.

THEOREM 17. *The weak neighborhood topology is metrizable as a norm-metric if and only if the space is finite dimensional.*

The non-metrizability of the infinite dimensional spaces follows from Lemma 3 and Theorem 6. That the finite dimensional spaces may be normed follows from Tychonoff's result [**6**, p. 769] that every finite dimensional linear topological space is homeomorphic to Euclidean space of the same dimension.

The situation with regard to metrizability of weak neighborhood topologies is different from that in the weak sequential topologies, for it is well known that in the space l_1 the weak sequential topology is equivalent to the norm topology [cf. **1**, p. 139]. One easily concludes then that in the weak neighborhood topology for l_1 there must exist points which are limit points without being sequential limit points.[11]

It is interesting to note that Hilbert space is also of the first category with its weak sequential topology. Let $\{\varphi_n\}$ be an orthonormal set. Then, since the Fourier coefficients $(f, \varphi_n) \to 0$ for all $f \in \mathfrak{H}$, φ_n converges weakly to θ and $3r\varphi_n + f$ converges weakly to f. But, if $S_r = E_f[\| f \| \leqq r]$, $3r\varphi_n + f$ non-ϵS_r if $f \in S_r$ so that each element of S_r is a limit point of a sequence not in S_r, i.e., S_r is non-dense. The rest of the proof follows as in Theorem 15.

11. **Linear transformations in weak normed spaces.** In this section we return to the discussion of the relation between bounded sets and continuity of additive transformations. The theorem which will be proved has some interest in itself and also serves as an example to show that Theorem 3′ may hold even when the domain is not metrizable.

THEOREM 18. *If L_1 and L_2 are two normed spaces with their weak neighborhood topologies, any additive transformation $T(x)$ which takes bounded sets into bounded sets is continuous.*

If $R \subset L_1$ is bounded in the weak topology, there must exist for each linear continuous functional, $F(x)$, on L_1 a number $M(F, R) \geqq 0$ such that $| F(R) | \leqq M(F, R)$, for otherwise for some $F(x)$ on L_1 there would exist $x_n \in R$ and $\alpha_n \to 0$ such that $F(\alpha_n x_n)$ does not converge to zero. First it will be shown that if $\| x_n \| \to 0$, then $T(x_n) \to \theta$ weakly. For if $\| x_n \| \to 0$ there exist positive integers $k_n \to \infty$ such that $\| k_n x_n \| = k_n \| x_n \| \to 0$. Since $\| k_n x_n \| \to 0$, $k_n x_n \to \theta$ weakly and hence $\{k_n x_n\}$ is weakly bounded. From the hypothesis it follows that $\{T(k_n x_n)\}$ is also weakly bounded. By the remark above there will exist for each linear continuous F an $M(F) \geqq 0$ such that $| F(T(k_n x_n)) | \leqq M(F)$ or, using additivity of F and T, such that $| F(T(x_n)) | \leqq k_n^{-1} M(F) \to 0$, i.e., $T(x_n) \to \theta$ weakly. Now, from the fact that $\| x_n \| \to 0$ implies $T(x_n) \to \theta$ weakly, one may show that $\| x_n \| \to 0$ implies $\| T(x_n) \| \to 0$, i.e., $T(x)$ is continuous in the norm topologies. This will be proved by showing that there

[11] A direct construction of such a point has been given by Köthe [**11**, p. 118]. Let $R = E_x[\| x \| \geqq 1]$. Then θ can obviously not be a sequential limit point of R, but, since $L - R$ is bounded, $\mathfrak{P}(R, U)$ is not vacuous for any $U \in \mathfrak{U}$. Hence θ is a limit point of R.

exists $M \geqq 0$ such that $\| T(x) \| \leqq M \| x \|$. Suppose there exists no such M. Then for each integer m there exists $y_m \in L_1$ such that $\| T(y_m) \| \geqq m^2 \| y_m \|$. Let $z_m = \dfrac{y_m}{m \| y_m \|}$. Then $\| z_m \| \to 0$ and hence $T(z_m) \to \theta$ weakly. However,

$$\| T(z_m) \| = \frac{1}{m \| y_m \|} \| T(y_m) \| > m,$$

i.e., the set $\{T(z_m)\}$ is not of bounded norm. This is contradictory, for it is well known that a sequence can converge weakly to θ only if it is of bounded norm [1, p. 133, Theorem 1]. Hence the transformation $T(x)$ must be a bounded transformation and therefore continuous in the norm topologies. But then $T(x)$ must also be continuous in the weak neighborhood topologies. For if F is a linear continuous functional on L_2, then $F(T)$ is a linear continuous functional on L_1. Hence, if $V(F_1, \cdots, F_n; \delta)$ is an arbitrary neighborhood in L_2, $U(F_1(T), \cdots, F_n(T); \delta)$ is a neighborhood in L_1 and has the property that $T(U) \subset V$, i.e., $T(x)$ is continuous in the weak neighborhood topologies.

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